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Journal of Geometry and Physics 56 (2006) 611–622
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Noncommutative differential calculus for Moyal subalgebras

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Received 24 November 2004; received in revised form 11 April 2005; accepted 13 April 2005
Available online 13 May 2005

Abstract

We build a differential calculus for subalgebras of the Moyal algebra on $\mathbb{R}^4$ starting from a redundant differential calculus on the Moyal algebra, which is suitable for reduction. In some cases we find a frame of one-forms which allows to realize the complex of forms as a tensor product of the noncommutative subalgebras with the external algebra Λ^*.

PACS: 02.40.Gh

MSC: 46L87

JGP SC: Noncommutative topology and geometry

Keyword: Noncommutative differential calculus

1. Introduction

In this paper we address the problem of building a differential calculus on a wide class of noncommutative algebras introduced in [1]. Those are inequivalent infinite-dimensional $*$-

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0393-0440/$ – see front matter
doi:10.1016/j.geomphys.2005.04.006

algebras in one-to-one correspondence with subalgebras of the Moyal algebra on $\mathbb{R}^4$, which all share the same commutative limit, namely the algebra of functions on $\mathbb{R}^3$. Following some ideas of Segal contained in [2], where he defines a *Quantized Differential Calculus* for the algebra of operators of quantum mechanics, we build a differential calculus based on the existence of a sufficient number of derivations. The algebras we are interested in are subalgebras of a bigger one, therefore, in our approach, an important point is how to infer a differential calculus for subalgebras from a given differential calculus on the big algebra. The problem is nontrivial in the noncommutative case, and of interest also in more general situations where we have morphisms between two algebras $\mathcal{A}$, $\mathcal{B}$, which could be, for example, the noncommutative analogues of the source and target space of field theories. In the commutative case, given M, N, a pair of differentiable manifolds with some $\phi : M \to N$, we know that the exterior derivative on the two spaces is connected by a pull-back:

$$\phi^*(d_N f) = d_M \phi^*(f), \tag{1.1}$$

where $f \in \mathcal{F}(N)$ while $\phi^*(f) \in \mathcal{F}(M)$. But, if the commutative algebras of functions $\mathcal{F}(M), \mathcal{F}(N)$ are replaced by the noncommutative algebras $\mathcal{A}, \mathcal{B}$ with some $\psi : \mathcal{B} \to \mathcal{A}$ the relation between the differential calculi on the two algebras is not obvious a priori. Can we use the differential calculus on $\mathcal{A}$ to define a differential calculus on $\mathcal{B}$ as in Eq. (1.1)? As we shall see, this is in general not possible, essentially because derivations in the noncommutative case are not a $\mathcal{A}$-module, namely we cannot multiply them by elements of $\mathcal{A}$ so that they remain derivations. This will affect the exterior derivative d. In other words, if we are given a basis of one-forms and an algebra of derivations for the noncommutative algebra, we may still write d as $d = \theta^a X_a$ but it is not true in general that we can perform a change of bases both for one-forms and derivations such that the same exterior derivative d is also equal to some $\alpha^a Y_a$. Indeed, once we have performed the change of basis for the one-forms (which we can do, the one-forms being a $\mathcal{A}$-module) we cannot rearrange the derivations in order that they stay derivations, apart from multiplying them by numbers or elements in the centre of $\mathcal{A}$. The main point of the paper is therefore the construction of a differential calculus for subalgebras of the Moyal algebra on R^4, $\mathcal{M}_\theta$, starting from the definition of a differential calculus on the Moyal algebra which is suitable to be reduced.

2. Differential calculus for (noncommutative) associative algebras

For an associative algebra a differential calculus can always be defined algebraically, once a Lie algebra of derivations, $\mathcal{L}$, is given (see for example [2,3]). A one-form α is a linear map from $\mathcal{L}$ to $\mathcal{A}$. An exterior derivative d is defined as

$$d\alpha(X, Y) = \rho(X)(\alpha(Y)) - \rho(Y)(\alpha(X)) - \alpha([X, Y]). \tag{2.1}$$

If $\rho : \mathcal{L} \to \mathrm{Der}(\mathcal{A})$ is a Lie algebra homomorphism, then $d^2 = d \circ d$ is zero. Higher forms are defined as skew-symmetric multilinear maps from $\mathcal{L}$ to the associative algebra $\mathcal{A}$. Thus, to define a differential calculus on a noncommutative algebra, we need to choose a set of derivations, that have to be independent and sufficient, and a representation of $\mathcal{L}$ on $\mathcal{A}$. (A set of derivations is said to be sufficient when the only elements which are annihilated by all of them are in the centre of the algebra.) That is, we need $\mathcal{L}$, ρ such that

$$\rho(X)(f * g) = (\rho(X)f) * g + f * (\rho(X)g), \quad X \in \mathcal{L}, \ f, g \in \mathcal{A}, \tag{2.2}$$

where $*$ is the noncommutative product in $\mathcal{A}$. Assuming such structures are given, the first step for the construction of a differential calculus is the identification of zero forms with the algebra itself

$$\Omega^0 = \mathcal{A}. \tag{2.3}$$

Then the exterior derivative is implicitly defined by

$$df(X) = \rho(X)f. \tag{2.4}$$

It automatically verifies the Leibnitz rule because $\rho(X)$, $X \in \mathcal{L}$ are $*$-derivations

$$d(f * g)(X) = (\rho(X)f) * g + f * (\rho(X)g), \tag{2.5}$$

moreover,

$$d^2 = 0, \tag{2.6}$$

because the $*$-derivations $\rho(X)$, $X \in \mathcal{L}$ close a Lie algebra. The second step consists in defining Ω^1 as a left $\mathcal{A}$ module that is

$$gd(X) = g * (\rho(X)f). \tag{2.7}$$

Analogously, we can define a right $\mathcal{A}$ module. Because of noncommutativity they are not the same, but we can always express one in terms of the other. Thus, we consider left modules from now on. To construct Ω^2 we use (2.1) and (2.6). We have

$$df \diamond dg(X_\mu, X_\nu) = df(X_\mu) * df(X_\nu) - df(X_\nu) * df(X_\mu), \tag{2.8}$$

where we have indicated with $\diamond$ the product of forms. Because of noncommutativity:

$$df \diamond dg \neq -dg \diamond df. \tag{2.9}$$

In a similar way to Ω^1, Ω^2 is defined as a left $\mathcal{A}$ module with respect to the $*$ multiplication:

$$fdg \diamond dh(X_\mu, X_\nu) = f * dg(X_\mu) * dh(X_\nu) - f * dg(X_\nu) * dh(X_\mu). \tag{2.10}$$

Higher Ω^p are built along the lines of the commutative case.

3. A differential calculus for the Moyal algebra

The simplest and mostly studied noncommutative algebra is the Moyal algebra. This is a deformation of the algebra of functions on $\mathbb{R}^{2n}$, $(\mathcal{F}(\mathbb{R}^{2n}), \cdot)$ into the noncommutative algebra $(\mathcal{M}, *_\theta)$ where $*_\theta$ is the Moyal product [4,5] and θ the noncommutativity parameter. The zero-th order in θ yields back the ordinary commutative product, while the first order is the Poisson bracket which we assume for simplicity the canonical one. Different (nondegenerate) Moyal products on $\mathbb{R}^4$ are in principle associated with an invertible antisymmetric matrix Θ_{ij} which, with a change of coordinates, can be expressed in the canonical form:

$$\Theta = \begin{pmatrix} 0 & 0 & -\theta_1 & 0 \\ 0 & 0 & 0 & -\theta_2 \\ \theta_1 & 0 & 0 & 0 \\ 0 & \theta_2 & 0 & 0 \end{pmatrix} \tag{3.1}$$

with $\pm\theta_i$ the eigenvalues of Θ. A simple rescaling can then equate $\theta_1 = \theta_2 = \theta$. In this setting, the Moyal product $f \star_\theta g$ of two Schwartz functions f, g on $\mathbb{R}^4$ is defined by

$$f *_\theta g(u) := \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} L^\theta(u, v, w)\, f(v) g(w)\, d\mu^\theta(v)\, d\mu^\theta(w), \tag{3.2}$$

where $u := (q, p)$; θ is a positive real parameter; $d\mu^\theta(v) := (\pi\theta)^{-4}\, d\mu(v)$. The integral kernel L^θ is given by

$$L^\theta(u, v, w) := \exp\left(\frac{2i}{\theta}(uJv + vJw + wJu)\right), \tag{3.3}$$

where J denotes the antisymmetric matrix:

$$J := \begin{pmatrix} 0 & \mathbb{1}_2 \\ -\mathbb{1}_2 & 0 \end{pmatrix} \tag{3.4}$$

with $\mathbb{1}_2$ the 2×2 identity matrix. What is properly defined as the Moyal algebra is $\mathcal{M}_\theta := \mathcal{M}_L(\mathbb{R}^4_\theta) \cap \mathcal{M}_R(\mathbb{R}^4_\theta)$ where $\mathcal{M}_L(\mathbb{R}^4_\theta)$, the left multiplier algebra, is defined as the subspace of tempered distributions that give rise to Schwartz functions when left multiplied by Schwartz functions; the right multiplier algebra $\mathcal{M}_R(\mathbb{R}^4_\theta)$ is analogously defined. For more details we refer to the appendix in [1] and references therein. In the present article we shall think of $\mathcal{M}_\theta$ as the algebra of $*$-polynomial functions in q_i, p_i, properly completed. Its commutative limit, $\mathcal{F}(\mathbb{R}^4)$, is the commutative multiplier algebra $\mathcal{O}_M(\mathbb{R}^4)$, the algebra of smooth functions of polynomial growth on $\mathbb{R}^4$ in all derivatives [6]. To define a differential calculus in the constructive way described in the previous section we need derivations. The $\mathcal{M}_\theta$ are normal spaces of distributions, and all their derivations are inner. Therefore, we turn our attention to groups of automorphisms of $\mathcal{M}_\theta$. A relevant one is the inhomogeneous symplectic group ISp(4, $\mathbb{R}$), constituted by translations and real symplectic transformations of $\mathbb{R}^4$.[1] As we will see below in more detail, it induces derivations both for the commutative algebra $\mathcal{F}(\mathbb{R}^4)$ and the Moyal algebra $\mathcal{M}_\theta$. In facts its Lie algebra is the maximal algebra of derivations with this property. Moreover, although it is not minimal (the subalgebra of translations would suffice) it generates the whole algebra of polynomial functions, once we represent its generators as quadratic-linear functions in $\mathbb{R}^4$.

The group Sp(4, $\mathbb{R}$) consists of elements g for which $g^t Jg = J$; this implies for the Lie algebra generators that $M^t J + JM = 0$, with

$$[M_a, M_b] = C^c_{ab} M_c \tag{3.5}$$

[1] Note however that smaller Moyal algebras can be chosen, such that the inhomogeneous symplectic algebra acts as outer derivations on them [6]. The choice of such big algebras in the present paper is motivated by the fact that they contain all polynomials.

and C^c_{ab} the structure constants of the symplectic algebra. The Lie algebra is realized in terms of vector fields on $\mathbb{R}^4$ by

$$Y_a = -M^{\mu}_{a\nu} u^{\nu} \frac{\partial}{\partial u^{\mu}}. \tag{3.6}$$

To the symmetric matrices $B_a = -JM_a$ it is associated a set of quadratic functions on $\mathbb{R}^4$:

$$y_a = \frac{1}{2} u^t B_a u. \tag{3.7}$$

They define a realization of the symplectic algebra as a Poisson algebra with respect to the canonical Poisson bracket $\{q_i, p_j\} = \delta_{ij}$:

$$\{y_a, y_b\} = C^c_{ab} y_c. \tag{3.8}$$

The inhomogeneous sector of the Lie algebra of $isp(4,\mathbb{R})$ is represented by linear functions. Therefore, the whole inhomogeneous symplectic algebra, $isp(4,\mathbb{R})$, may be realized as a Poisson algebra on $\mathbb{R}^4$ with generators a set of quadratic-linear functions of q, p. A possible choice for the generators is

$$\begin{aligned}
&y_1 = \frac{1}{2}(q_1 q_2 + p_1 p_2), \qquad y_2 = \frac{1}{2}(q_1 p_2 - q_2 p_1), \quad y_3 = \frac{1}{4}(q_1^2 + p_1^2 - q_2^2 - p_2^2),\\
&y_4 = \frac{1}{4}(q_1^2 + p_1^2 + q_2^2 + p_2^2), \qquad y_5 = \frac{1}{4}(q_1^2 + q_2^2 - p_1^2 - p_2^2),\\
&y_6 = \frac{1}{2}(q_1 p_1 + q_2 p_2), \qquad y_7 = \frac{1}{2}(q_1 p_2 + q_2 p_1), \qquad y_8 = \frac{1}{2}(q_1 p_1 - q_2 p_2),\\
&y_9 = \frac{1}{2}(q_1 q_2 - p_1 p_2),
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
&y_{10} = \frac{1}{4}(q_1^2 - q_2^2 - p_1^2 + p_2^2), \qquad y_{11} = q_1, \qquad y_{12} = q_2,\\
&y_{13} = p_1, \qquad y_{14} = p_2.
\end{aligned} \tag{3.10}$$

We have (latin indices now run from 1 to 14):

$$\{y_a, y_b\} = C^c_{ab} y_c, \tag{3.11}$$

with C^c_{ab} the structure constants of the whole $isp(4,\mathbb{R})$. Thus, the generators of the Lie algebra $isp(4, \mathbb{R})$, act as inner derivations in $\mathcal{F}(\mathbb{R}^4)$ with

$$\rho(M_a)(f) = Y_a(f) = \{y_a, f\}. \tag{3.12}$$

Let us notice that the vector fields Y_a are the Hamiltonian vector fields associated to the functions y_a; therefore, to the linear functions q_i, p_i $i = 1, 2$ we associate the vector fields $\partial/\partial p_i, -\partial/\partial q_i$, respectively. The algebra of quadratic-linear functions of q, p is also closed with respect to the Moyal product: using the asymptotic development, which becomes exact when at least one of the two elements of the product is quadratic-linear, it is possible to show [1] that the product of two such functions is still a function of $\{y_a\}$:

$$y_a * y_b = f(\{y_c\}). \tag{3.13}$$

Moreover, the Moyal bracket or $*$-commutator essentially coincides with the Poisson bracket (3.11):

$$[y_a, y_b]_* = i\theta C^c_{ab} y_c. \tag{3.14}$$

Thus, the generators of the Lie algebra $isp(4,\mathbb{R})$, act as inner derivations in $\mathcal{M}_\theta$ as well, with

$$\rho(M_a)(f) = [y_a, f]_*, \quad f \in \mathcal{M}_\theta \tag{3.15}$$

with the Leibniz rule trivially satisfied. Thus, $isp(4,\mathbb{R})$ plays the double role of generating $\mathcal{M}_\theta$ and furnishing $*$-derivations; moreover, this is true in the commutative limit as well. According to the general procedure outlined in the previous section, once we have derivations we can define an exterior derivative d and construct a differential calculus on $\mathcal{M}_\theta$ which is certainly not minimal, but has interesting properties. The idea we want to pursue, which will be developed in the next section, recalls very much the construction of a differential calculus on the algebra of $N \times N$ matrices described by Madore [7]. There, a redundant calculus is constructed which is what is needed to define differential calculi for different subalgebras of Mat(N). Here, after identifying many interesting subalgebras of $\mathcal{M}_\theta$ we will define a differential calculus for each of them, the main difference with the previous case being that our algebras are realized as operators on infinite dimensional space.

4. A differential calculus for subalgebras

The algebra $\mathcal{M}_\theta$ has interesting subalgebras, which we indicate generically with $\mathcal{B}$, which share the same commutative limit, $\mathcal{F}(\mathbb{R}^3)$. Therefore, we regard them as different deformations of $\mathcal{F}(\mathbb{R}^3)$, each of them with its own $*$-product. Those subalgebras are polynomially generated by three-dimensional subsets of the quadratic linear functions y_μ given by (3.10). It can be shown that they are in one to one correspondence with three-dimensional Lie algebras [8] which they realize both as Poisson algebras [8] and as $*$ algebras [1]. We briefly review the procedure followed in [1] for the convenience of the reader. Consider first the identification $\mathcal{G}^* \equiv \mathbb{R}^3$, where $\mathcal{G}^*$ is the dual algebra of some three dimensional Lie algebra. It is known that all three dimensional algebras can be classified and a Poisson realization can be given once the generators of a Lie algebra are identified with the linear functions on the dual [9,8]. In normal form we have

$$\{x, y\} = cw + hy, \qquad \{y, w\} = ax, \qquad \{w, x\} = by - hw, \tag{4.1}$$

where a, b, c, h are real parameters characterizing the algebras and satisfying the condition $ah = 0$. Choosing appropriately the parameters we reproduce all the three-dimensional Lie algebras.

Consider now $\mathbb{R}^4$ with the canonical symplectic structure given by the Poisson brackets

$$\{q_i, p_j\} = \delta_{ij},$$

associated to the symplectic form:

$$\omega = dq_1 \wedge dp_1 + dq_2 \wedge dp_2.$$

It is possible to find symplectic realizations $\pi : \mathbb{R}^4 \to \mathcal{G}^* \equiv \mathbb{R}^3$, for $\mathcal{G}^*$ dual to any three-dimensional Lie algebra, $\mathcal{G}$. We express π through the change of variables π^* that pulls smooth functions on $\mathbb{R}^3$ back to smooth functions on $\mathbb{R}^4$. All that one has to do is to find three independent functions f_1, f_2, f_3 on $\mathbb{R}^4$ whose corresponding canonical brackets have the required form (4.1). The Poisson map π is *not* required to be onto, nor a submersion, that is to say, to arise from a regular foliation of $\mathbb{R}^3$.

Several π-maps were constructed in [8], under the name of (generalized) classical Jordan–Schwinger maps. Although many realizations are possible, it turns out that it is always possible to find a realization for every $\mathcal{G}$ in terms of quadratic-linear functions on $\mathbb{R}^4$, namely as a Poisson subalgebra, $\mathcal{B}$, of the algebra *isp*(4) given by (3.10).

Less obvious is that the subalgebras are also closed under the induced $*$-product. We have indeed (now Latin indices run from 1 to 3):

$$y_i * y_j = f(\{y_i\}) \qquad [y_i, y_j]_* = i\theta\{y_i, y_j\}. \tag{4.2}$$

That is, for each set of generators we observe that:

- they generate polynomially a noncommutative subalgebra of $\mathcal{M}_\theta$, say $\mathcal{B}_\mathcal{G}$;
- they close the Lie algebra $\mathcal{G}$, which is a subalgebra of *isp*(4,$\mathbb{R}$), both with respect to the Moyal bracket and to the Poisson bracket;
- this implies that $\mathcal{G}$ acts on $\mathcal{B}_\mathcal{G}$ in terms of inner $*$-derivations:

$$\rho(M_i)f = [y_i, f]_*, \quad M_i \in \mathcal{G}, \quad f \in \mathcal{B}. \tag{4.3}$$

A detailed account of all the subalgebras of (3.10) and related star products is contained in [1]. We shall only recall that there are essentially two families of subalgebras. Those that we call of type A in the cited article, which are defined by the property of being the commutant of a certain function in the list (3.10), which can be identified as the Casimir function, it corresponding exactly to the Casimir of the associated Lie algebra. To this class belongs the example we study below. To the other broad class belong the so called type B algebras, that is, algebras defined through a Casimir one-form which is not exact. Among them, an interesting case is the k-Minkowski algebra which describes a deformed $2 + 1$ Minkowski space. The differential calculus that we construct is essentially different for the two cases. We will see that for type A algebras it is possible to find a frame of one-forms which behave as in the commutative case, whereas for type B algebras this is not possible.

As a guiding example we shall refer, when needed, to the type A subalgebra generated by y_1, y_2, y_3 as in Eq. (3.10). As a Poisson algebra this is easily seen to be isomorphic to *su*(2). More precisely, it is the commutant of y_4. The induced star product is in that case

$$y_j *_{su(2)} f(y_i) = \left\{ y_j - \frac{i\theta}{2}\epsilon_{jlm} y_l \partial_m - \frac{\theta^2}{8}\left[(1 + y_k\partial_k)\partial_j - \frac{1}{2} y_j\,\partial_k\partial_k\right]\right\} f(y_i). \tag{4.4}$$

The nonlocality of this product is evident once we observe that $y_i * y_i = y_i^2 - (1/8)\theta^2$ (no sum over repeated indexes). The algebra generators may be represented in terms of creating and annihilating operators acting on the usual Hilbert space of the two dimensional harmonic oscillator, with basis the cartesian kets $|n_1 n_2\rangle$. The sum $n_1 + n_2$ is constant, it being the eigenvalue of y_4, which commutes with the whole algebra and represents the Hamiltonian of the system of oscillators. Therefore, changing basis to the $\{$Hamiltonian + angular momentum$\}$ basis, it is possible to see that for each value of the angular momentum there is a representation of $su(2)$. The noncommutative algebra of functions of R^3 therefore reduces to a set of finite dimensional algebras, receptacles for representations of $su(2)$. Each reduced block is the algebra of a fuzzy sphere [10] in the oscillator representation. Therefore, the three dimensional space is "foliated" as a set of fuzzy spheres of increasing radius. We can give a geometric interpretation of the new star product. Note that, with the exception of the zero orbit, the orbits of the Hamiltonian system associated to y_4 are circles. Functions of (y_1, y_2, y_3) correspond here to functions of (q_1, q_2, p_1, p_2) that remain invariant on those orbits. We are thus identifying R^3 to the foliation of R^4 by those trajectories. The orbits rest on spheres in R^4. One circle and only one passes through each point different from 0. The corresponding maps $S^3 \to S^2$ are Hopf fibrations.

A differential calculus on each $\mathcal{B}_\mathcal{G}$ is straightforward to define along the same lines of the previous section. This is the natural reduction of the differential calculus on $\mathcal{M}_\theta$ to the subalgebras $\mathcal{B}_\mathcal{G}$. In particular the exterior derivative may be defined as

$$d_\mathcal{B} f(M_i) = [y_i, f]_* \tag{4.5}$$

with $f, y_i \in \mathcal{B}_\mathcal{G}$, $M_i \in \mathcal{G}$. If $\phi : \mathcal{B}_\mathcal{G} \to \mathcal{M}_\theta$ is the embedding in the Moyal algebra we have

$$\phi(d_\mathcal{B} f) = d_\mathcal{M} \phi(f) \tag{4.6}$$

that is, the differential calculus we have defined on the Moyal algebra induces a differential calculus on subalgebras. The condition for that to be possible in the noncommutative case is that the derivations we have chosen for $\mathcal{M}_\theta$ be 'adapted' to those of $\mathcal{B}$. Had we chosen as an algebra of derivations just the translations, which is what gives the minimal differential calculus on the Moyal algebra, Eq. (4.6) would not have been true. This justifies a posteriori our choice of such a big calculus for $\mathcal{M}_\theta$.

4.1. Frame of one-forms

For each fixed subalgebra $\mathcal{B}_\mathcal{G}$ the set of $\{dy_i\}$ certainly constitutes a system of generators of $\Omega^1(\mathcal{B}_\mathcal{G})$ but it is not the most convenient one. Because of noncommutativity we have indeed $f(y_i)dy_j \neq dy_j f(y_i)$. A better system of generators for Ω^1 would be one-forms which are dual to the derivations $\rho(M_i)$. Finding a frame of one-forms for the subalgebras $\mathcal{B}_\mathcal{G}$ is not always possible. In facts, there is no solution in all the cases where the algebra has neither a centre nor a unity (type B algebras of [1]), but we will show that a solution exists for the subalgebra considered above.

Once we have found the frame, the construction of the differential calculus follows very closely the construction of Madore [7] for finite-dimensional (matrix) algebras, although ours is not finite-dimensional. Differential calculi constructed in this way depend on the algebra of derivations one has chosen. We will show at the end of the section how it can be made independent on derivations and formulated in terms of a one-form which recalls the Dirac operator of Connes differential calculus [11].

For a generic subalgebra $\mathcal{B}_{\mathcal{G}}$ the system to be solved is

$$(\alpha^i)(M_j) \in Z(\mathcal{B}_{\mathcal{G}}), \tag{4.7}$$

where $i = 1, \dots 3$, M_i are the generators of the Lie algebra of $*$-derivations of $\mathcal{B}_{\mathcal{G}}$ and $Z(\mathcal{B})$ is the centre of $\mathcal{B}$. Here we have slightly modified the definition of dual frame, because our algebras have no unity. If the centre is trivial (4.7) has no solutions. Therefore, the problem is meaningful only for type A subalgebras.

Let us consider the algebra $\mathcal{B}_{su(2)}$. To the centre belong all functions of $y_1^2 + y_2^2 + y_3^2 = [(q_1^2 + q_2^2 + p_1^2 + p_2^2)/4]^2$. Since a generic one-form may be written as $\sum_a f_a dg_a$ it can be easily seen that there are no solutions which can be expanded in the basis $\{dy_i\}$. Therefore, we write the dual forms as

$$\alpha^i = f_1^i dq_1 + f_2^i dq_2 + f_3^i dp_1 + f_4^i dp_2 \tag{4.8}$$

where f^i are functions in R^4 and the one-forms dq_i, dp_i are defined by (2.4). By means of Eqs. (4.5) and (4.7) becomes then

$$f^i * \frac{i\theta}{2} A = z\delta^{ij} e_j \tag{4.9}$$

where f^i is a row vector, A is the 3×4 matrix

$$\begin{pmatrix} -p_2 & q_2 & -p_1 \\ -p_1 & -q_1 & p_2 \\ q_2 & p_2 & q_1 \\ q_1 & -p_1 & -q_2 \end{pmatrix}, \tag{4.10}$$

z is an element in the centre and e_j is the row vector $(0, \dots, 1_j, 0, \dots)$. Since the algebra $\mathcal{B}_{su(2)}$ has a centre, solutions to (4.7) are in principle defined up to one-forms in the kernel of $\{M_1, M_2, M_3\}$ (‘Casimir one-forms’). Therefore, to solve the problem we enlarge the algebra of derivations introducing the one associated to the generator M_4, which commutes with all the others. This is represented by the quadratic function y_4 in the list (3.10), which commutes with all the elements of our algebra. We look then for a one-form, α^4, dual to this auxiliary derivation. If existing, it will be a Casimir one-form. The system to be solved becomes now

$$f^\mu * \frac{i\theta}{2} A = z\delta^{\mu\nu} e_\nu \tag{4.11}$$

where, with an obvious extension of the notation, the index μ runs from 1 to 4 and A is the square matrix:

$$\begin{pmatrix} -p_2 & q_2 & -p_1 & -p_1 \\ -p_1 & -q_1 & p_2 & -p_2 \\ q_2 & p_2 & q_1 & q_1 \\ q_1 & -p_1 & -q_2 & q_2 \end{pmatrix}. \tag{4.12}$$

We need therefore a right $*$-inverse, that is a matrix B such that $A * B = z$. In general we are not guaranteed that a matrix with noncommuting entries have an inverse in the sense specified above; in this case it exists (indeed, it is possible to define a $*$-determinant, which is non-zero and central). Notice that in the commutative limit $\det A = 0$, that is the matrix A is degenerate. The solution for the frame of one-forms is finally

$$\begin{aligned} \alpha^1 &= C\left[i(p_2 dq_1 + p_1 dq_2 - q_2 dp_1 - q_1 dp_2) - \frac{2}{\theta} y_1 \beta\right], \\ \alpha^2 &= C\left[i(q_1 dq_2 - q_2 dq_1 - p_2 dp_1 + p_1 dp_2) - \frac{2}{\theta} y_2 \beta\right], \\ \alpha^3 &= C\left[i(p_1 dq_1 - p_2 dq_2 - q_1 dp_1 + q_2 dp_2) - \frac{2}{\theta} y_3 \beta\right], \end{aligned} \tag{4.13}$$

$$\alpha^4 = C\left[\frac{2}{\theta} y_4 \beta\right], \tag{4.14}$$

where $\beta = (q_1 dq_1 + q_2 dq_2 + p_1 dp_1 + p_2 dp_2)$, θ is the noncommutativity parameter and C is a normalization constant. The one-form β is in the kernel of the algebra of derivations generated by M_1, M_2, M_3 (notice however that it is not equal to $d(q_1^2 + q_2^2 + p_1^2 + p_2^2)/2$), therefore, it is ineffective as long as we are concerned with the subalgebra generated by $\{y_1, y_2, y_3\}$. The differential calculus which we have induced on $\mathcal{B}_{su(2)}$ is three-dimensional and generated by the frame of one-forms $\alpha_1, \alpha_2, \alpha_3$. Notice that, up to the one-form β, these are exactly the dual one-forms of left invariant vector fields on the group manifold $SU(2)$ when immersed in R^4. In facts, from (4.3), in the commutative limit the three derivations go into the vector fields

$$\begin{aligned} Y_1 &= D\left(p_2 \frac{\partial}{\partial q_1} + p_1 \frac{\partial}{\partial q_2} - q_2 \frac{\partial}{\partial p_1} - q_1 \frac{\partial}{\partial p_2}\right), \\ Y_2 &= D\left(-q_2 \frac{\partial}{\partial q_1} + q_1 \frac{\partial}{\partial q_2} - p_2 \frac{\partial}{\partial p_1} + p_1 \frac{\partial}{\partial p_2}\right), \\ Y_3 &= D\left(p_1 \frac{\partial}{\partial q_1} - p_2 \frac{\partial}{\partial q_2} - q_1 \frac{\partial}{\partial p_1} + q_2 \frac{\partial}{\partial p_2}\right), \end{aligned} \tag{4.15}$$

which can be recognized to be a basis of left invariant vector fields on the three-sphere. These are independent if we only allow numerical coefficients, but not as a module. In the noncommutative case they are independent because there is no module structure. Therefore, recalling the geometric interpretation we have given of the representation space of the $\mathcal{B}_{su(2)}$

algebra as a foliation into fuzzy spheres, we recover the known result that the tangent space to noncommutative two-spheres is three-dimensional and not two-dimensional.

As anticipated in the beginning of the section, the existence of a frame simplifies very much the construction of the differential calculus and makes it possible to model it on the existing differential calculi for finite-dimensional matrix algebras, thus allowing to recover many of the properties we have in that case. We will enumerate some of them. Because of their definition (4.7) fundamental forms verify

$$f\alpha^i = \alpha^i f. \tag{4.16}$$

Then $\Omega^1(\mathcal{B})$ is a free module of rank 3. Moreover, $\alpha^i \diamond \alpha^j = -\alpha^j \diamond \alpha^i$ which implies that forms of degree higher than 3 vanish.

From the same equation (4.7) we derive the Lie derivative:

$$0 = \mathcal{L}_{Y_i}\langle Y_j, \alpha^k\rangle = \langle(\mathcal{L}_{Y_i} Y_j), \alpha^k\rangle + \langle Y_j, \mathcal{L}_{Y_i}\alpha^k\rangle. \tag{4.17}$$

The Lie derivative of a derivation being just the Lie bracket we have then

$$\mathcal{L}_{Y_i}\alpha^k = \alpha^l \epsilon_{li}^k. \tag{4.18}$$

From the definition of exterior derivative (2.4) we find an important property of fundamental forms:

$$d\alpha^i = \frac{1}{2}\epsilon^i_{\ jk}\alpha^j \diamond \alpha^k, \tag{4.19}$$

which is the Maurer Cartan equation. The fundamental one-forms being graded-commutative we can construct the external algebra Λ^*, so that $\Omega^*(\mathcal{B}) = \mathcal{B} \otimes \Lambda^*$.

From the fundamental forms α_i we can construct a one-form in $\Omega^1(\mathcal{B})$:

$$\alpha = -y_i\alpha^i \tag{4.20}$$

in terms of which we can reexpress the exterior derivative df as

$$df = -[\alpha, f]. \tag{4.21}$$

Here, there is no explicit reference to derivations. The one-form α generates $\Omega^1(\mathcal{B}_{su(2)})$ as a bimodule.

The construction of $\Omega^1(\mathcal{B}_{su(2)})$ that we have presented in this section may be easily repeated for the subalgebra $\mathcal{B}_{su(1,1)}$. The other type A algebras may be obtained as contractions of either $\mathcal{B}_{su(2)}$, or $\mathcal{B}_{su(1,1)}$, therefore, it should be possible to generate for them a frame of one-forms through a contraction procedure.

5. Concluding remarks

In this paper we have addressed the problem of defining a differential calculus for noncommutative algebras possessing a sufficient number of $*$-derivations. To this purpose we have reviewed a procedure due to Segal to define a differential calculus for the algebra of operators of quantum mechanics, where the main ingredient was the existence of a Lie algebra of derivations. Inspired by an existing construction for matrix algebras due to Madore, we have found for a relevant case a frame of one-forms and discussed the commutative limit.

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